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On the Stable Distribution Function and a Fractional Stochastic Model for the Diffusion of the Brain Tumor Cancer

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ABSTRACT: In this note, we present a fractional stochastic model for the diffusion of the brain tumor cancer. The considered model is an extension of the deterministic glioma growth model. The stochastic process, which represents the solution of the considered model, is obtained in terms of the probability stable distribution function and in exact form. In the frame of such fractional stochastic models, we are able to study the growth models for tumor cells under the influence of random perturbations.

Keywords: Fractional stochastic diffusion model, Stable probability distribution function, Brain tumor cancer.

1. Introduction

Different types of dynamical systems and stochastic models of brain cancer progressions and treatments have already been constructed. They encompasses the invasive diffuse properties of the brain cancer and their growth rate. Following the model developed by Bergress and continued by James Mussary, [1-4], we complete their results by studying a factional growth model for tumor brain cells under the influence of random perturbations. We shall generalize the results in [5].

Let $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}_t}, \boldsymbol{P})$ be a filtered probability space and let $\{\boldsymbol{W(t)}, \boldsymbol{t} \geq \boldsymbol{0}\}$ be a standard Wiener process adapted to the filtration $(\boldsymbol{\mathcal{F}_t}, \boldsymbol{t} \geq \boldsymbol{0})$.

Consider the following fractional stochastic model:

$$B(x,t) = \varphi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\frac{a}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial B(x,s)}{\partial x}\right) + (\rho - K)B(x,s)\right]ds +$$

$$\frac{\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, B(x,s)dW(s), \qquad \textbf{(1.1)}$$

Where $0 < \propto \leq 1, \Gamma$ is the gamma function, $B(x,t)$ is the cell density at time t and radius x, a is the diffusion coefficient, expressed as cm^2 per day, K is the killing rate of tumor cells, $\varrho B(x,t)$ is the growth of tumor cells and σ is a constant, see [6-10].

It is assumed that φ is a given deterministic continuous function defined on an interval [0,L].

It is assumed also that the stochastic process $B(x,t)$ satisfies the boundary conditions:

$$B(0,t) = \beta(t),\ B(L,t) = \gamma(t),\ t \geq 0, \tag{1.2}$$

Where β, γ are stochastic processes. It is supposed that the stochastic process $\gamma(t)$ is independent of $W(t)$. It is supposed also that the process $\frac{d\gamma(t)}{dt}$ is measurable and bounded on the interval $[0,T]. T > 0$.

In section 2, we shall find exact formula for $B(x,t)$ and $E[(B(x,t)]$, where $E(X)$ is the expectation of the random variable X.

2-Exact formula for Brain cells

Let us simplify equation (1.1) by the substitution $u(x,t) = xB(x,t) - x\gamma(t)$. It is easy to get

$$u(x,t) + x\gamma(t) = x\varphi(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[a\frac{\partial^2 u(x,s)}{\partial x^2} + (\varrho - K)u(x,s)]ds +$$

$$\frac{\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(x,s)dW(s) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\varrho - \mathrm{K})x\gamma(s)ds + \frac{\sigma}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x\gamma(s)dW(s), \tag{2.1}$$

Notice that $u(0,t) = u(L,t) = 0$, $u(x,0) = x\varphi(x) - x\gamma(0)$.

We consider first the case when $\alpha = 1$.

Let v be the solution of the stochastic differential equation:

$$dv(x,t) = a\frac{\partial^2 v(x,t)}{\partial x^2}dt + (\varrho - \mathrm{K})v(x,t)dt + \sigma v(x,t)dW(t) - x\frac{d\gamma(t)}{dt}dt + \sigma x\gamma(t)dW(t), \tag{2.2}$$

Where $v(x,0) = x\varphi(x) - x\gamma(0), v(0,t) = v(L,t) = 0.$ (2.3)

Consider now the following stochastic differential equations:

$$dX_1(t) = \left[\frac{\sigma^2}{2}X_1(t) - (\varrho - \mathrm{K})X_1(t)\right]dt - \sigma X_1(t)dW(t), \tag{2.4}$$

$$dX_2(t) = \left[\frac{\sigma^2}{2}X_2(t) + (\varrho - \mathrm{K})X_2(t)\right]dt + \sigma X_2(t)dW(t). \tag{2.5}$$

The solutions of these two stochastic differential equations are given by:

$$X_1(t) = exp[-\{\sigma W(t) + (\varrho - K)t\}], \qquad X_2(t) = \exp[\sigma W(t) + (\varrho - K)t].$$

Set $v_1(x,t) = X_1(t)v(x,t)$ and applying the formula of Ito, we get:

$$dv_1(x,t) = X_1(t)dv(x,t) + v(x,t)dX_1(t) - \sigma^2[v(x,t) + x\gamma(t)]X_1(t)dt. \tag{2.6}$$

Substituting from (2.2) and (2.4) into (2.6), we get

$$dv_1(x,t) = \left[a\frac{\partial^2 v_1(x,t)}{\partial x^2} - \frac{\sigma^2}{2}v_1(x,t)\right]dt - \left[x\frac{d\gamma(t)}{dt} + \sigma^2 x\gamma(t)\right]X_1(t)dt + \sigma x\gamma(t)X_1(t)dW(t).$$

Thus:

$$\mathrm{d}v^*(x,t) = a\frac{\partial^2 v^*(x,t)}{\partial x^2}dt - e^{\frac{\sigma^2}{2}t}\left[x\frac{d\gamma(t)}{dt} + \sigma^2 x\gamma(t)\right]X_1(t)dt + e^{\frac{\sigma^2}{2}t}\sigma x\gamma(t)X_1(t)dW(t),$$
(2.7)

Where $v^*(x,t) = e^{\frac{\sigma^2}{2}t}\, v_1(x,t)$.

Notice that $v^*(x,t)$ **satisfies the following initial condition and boundary conditions:**

$$v^*(x,0) = x\varphi(x) - x\gamma(0), v^*(0,t) = v^*(L,t) = 0.$$
(2.8)

Let us solve the stochastic mixed problem (2.7) , (2.8).

Set $v^*(x,t) = \sum_{n=1}^{\infty} T_n(t) sin\frac{n\pi x}{L}$ **, where** $T_n(t) = \frac{2}{L}\int_0^L v^*(x,t) sin\frac{n\pi x}{L}dx$**.**

It is easy to get

$$\boldsymbol{dT_n(t) = -c_n T_n(t)dt + F_1(t)dt + F_2(t)dW(t),}$$

Where $\boldsymbol{F_1(t) = (-1)^n\frac{2}{n\pi}e^{\frac{\sigma^2}{2}t}\left[\frac{d\gamma(t)}{dt} + \sigma^2\gamma(t)\right]X_1(t)}$, $\boldsymbol{F_2(t) = (-1)^{n-1}\frac{2}{n\pi}e^{\frac{\sigma^2}{2}t}\sigma\gamma(t)X_1(t),}$

$c_n = \boldsymbol{a}\left(\frac{n\pi}{L}\right)^2$**. Thus** $\boldsymbol{T_n(t)}$ **is given by:**

$$\boldsymbol{T_n(t) = e^{-c_n t}T_n(0) + \int_0^t e^{-c_n(t-s)}F_1(s)ds + \int_0^t e^{-c_n(t-s)}F_2(s)dW(s),}$$

$\boldsymbol{T_n(0) = \frac{2}{L}\int_0^L x[\varphi(x) - \gamma(0)]sin\frac{n\pi x}{L}dx}$**. Consequently, the stochastic process** $\boldsymbol{v(x,t)}$ **is given by:**

$$\boldsymbol{v\{x,t) = e^{\frac{-\sigma^2 t}{2}}X_2(t)v^*(x,t).}$$

If $\boldsymbol{\gamma(t) = 0}$**, we get** $\boldsymbol{v(x,t) = e^{\frac{-\sigma^2 t}{2}}X_2(t)\sum_{n=1}^{\infty} e^{-c_n t}T_n(0)sin\frac{n\pi x}{L}}$**.**

Using our previous results [11-16], we can write:

$\boldsymbol{u(x,t) = \int_0^{\infty}\zeta_\alpha(\theta)v(x,t^\alpha\theta)d\theta}$**, where** $\boldsymbol{\zeta_\alpha(\theta)}$ **is the stable probability density function.**

Since $\boldsymbol{E\left[e^{\sigma W(t)}\right] = e^{\frac{\sigma^2}{2}t}}$ **, it follows that** $\boldsymbol{E[u(x,t)] = \sum_{n=1}^{\infty}\int_0^{\infty}\zeta_\alpha(\theta)\, e^{-c_n t^\alpha\theta}e^{(\varrho-\mathrm{K})t^\alpha\theta}T_n(0)sin\frac{n\pi x}{L}d\theta}$**.**

3- A fractional Stochastic Cauchy problem

We shall solve equation (1.1), for $\boldsymbol{x\epsilon(-\infty,\infty)}$ **,** $\boldsymbol{t > 0}$**.**

Let v **be the solution of the equation:**

$$\boldsymbol{dv(x,t) = a\frac{\partial^2 v(x,t)}{\partial x^2}dt + (\varrho - \mathrm{K})v(x,t)dt + \sigma v(x,t)dW(t), \qquad v(x,0) = x\varphi(x).}$$

Similar to section 2, one gets

$$dv^*(x,t) = a\frac{\partial^2 v^*(x,t)}{\partial x^2}dt, \text{ where } v^*(x,t) = e^{\frac{\sigma^2}{2}t}X_1(t)v(x,t).$$

Consequently the stochastic process $v(x,t)$ is given by

$$v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)y\varphi(y)dy,$$

Where $G(x,t) = \frac{1}{\sqrt{4\pi at}}e^{\frac{-\sigma^2 t}{2}}X_2(t)e^{\frac{-x^2}{4at}}$. Thus the stochastic process $u(x,t)$ is given by:

$$u(x,t) = \int_0^{\infty}\int_{-\infty}^{\infty} \zeta_\alpha(\theta)G(x-y,t^\alpha\theta)y\varphi(y)dyd\theta.$$

It is easy to see that $E[u(x,t)] = \int_0^\infty \int_{-\infty}^{\infty} \zeta_\alpha(\theta)G^*(x-y,t^\alpha\theta)y\varphi(y)dyd\theta$,

Where $G^*(x,t) = \frac{1}{\sqrt{4\pi at}}e^{\frac{-x^2}{4at}}e^{(\varrho-K)t}$. See [17-24].

4- Conclusion

Some stochastic mathematical models of brain cancer are studied. Fractional stochastic models are also considered. We have studied the fractional stochastic Burgess model. The solutions of stochastic mixed problem and stochastic Cauchy problem are obtained.

CONFLICTS OF INTEREST

There are no conflicts to declare.

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