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A Characterization of Strong and Weak convergence in Fuzzy Metric Spaces

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ABSTRACT: In this paper, we attempt to introduce the concept of fuzzy metric space and some of its properties, and we investigate the strong and weak convergence in fuzzy metric Spaces. Despite uncertainty in fuzzy random variables, crisp metrics have always been used. Here, we use the strong law of large numbers for fuzzy random variables in the fuzzy metric space for the bootstrap mean. Then the problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. In particular, and by modifying a definition of fuzzy metric space given by Kramosil and Michalek, Georgeand Veeramani have introduced and studied the following interesting notion of a fuzzy metric space. A fuzzy metric space is $(P_F(X), d_F)$ such that $P_F(X)$ is a set and d_F is a function defined on $d_F : P_F(X) \times P_F(X) \to [0, 1]$ satisfying certain axioms and d_F is called a fuzzy metric in $P_F(X)$. Our main result is the following characterization: a fuzzy metric space is strongly complete if and only if every nested sequence of close subsets which has strong fuzzy diameter zero has a singleton intersection. Moreover, the standard fuzzy metric is studied as a particular case. Finally, we developed ideas that many of known strong and weak convergence theorems can easily be derived from the fuzzy metric Spaces.

Keywords: Fuzzy metric space, Limit theorems, Cauchy sequence, fuzzy diameter, strong and weak convergence, Random set, Fuzzy random variable.

1. Introduction

This paper is organized as follows fuzzy metric spaces, strong fuzzy metric spaces and fuzzy linear normed spaces are defined and some examples are given to show the existence of these kinds of spaces; In the convergence of sequences of fuzzy points and the completeness of induced fuzzy metric spaces are considered; In 1965, the concept of fuzzy Sets was introduced by L. A. Zadeh [1]. Since then many authors have expansively developed the theory of fuzzy Sets and applications–Especially, Deng [2], Erceg [3], Kavela and Seokkala [4],kramosil and Michalek[5] has introduced the concept of fuzzy metric spaces in different ways. How to define fuzzy metric is one of the fundamental problems in fuzzy mathematics which is wildly used in fuzzy optimization and pattern recognition. There are two approaches in this field such as one is using fuzzy numbers to define metric in ordinary spaces, firstly proposed by Kaleva, following which fuzzy normed spaces, fuzzy topology induced by fuzzy metric spaces, fixed point theorem and other properties of fuzzy metric spaces are studied by a few researchers, see for instance, Felbin (1992)[6], George (1994)[7], Gregori (2000)[8], Hadzic (2002)[9] etc. The other one is using real numbers to measure the distances between fuzzy sets. Recently, many authors have also studied the fixed point theory in these spaces. For more details see [10]. In this work, we obtain sufficiently many of fuzzy metric spaces. Results of these researches have been applied to many practical problems in fuzzy environment. While, usually, different measures are used in different problems in other words, there does not exist a uniform measure that can be used in all kinds of fuzzy environments. Therefore, it is still interesting to find some kind of new fuzzy measure such that it may be useful for solving some problems in fuzzy environment. The attempt of the present paper is using fuzzy points defined on the real-valued space R to measure the distances between fuzzy points, which is consistent with the theory of fuzzy linear spaces in the sense of Xia and Guo (2003) [11] and hence more similar to the classical metric spaces. As in the classical case, the concept of Cauchy sequence plays a crucial role. Several concepts of Cauchy sequence in this context have been given in the literature [12]. In particular, and in a natural way, Gregori and Miñana [13] introduced and studied a concept of Cauchy sequence, named strong Cauchy sequence. Nevertheless, up to now, the two more used concepts of Cauchy sequence in fuzzy fixed point theory are due to M. Grabiec [14] and George and Veeramani [15]. Here at present, we discussed of strong and weak convergence in fuzzy metric Spaces.

2. Preliminaries

Fuzzy metric space:

Let X be any non-empty set. Then the mapping $d_F : P_F(X) \times P_F(X) \to S_F{}^{+}(R)$ is said to be a fuzzy metric space if for any $\{(x, \lambda), (y, \gamma), (z, \rho)\} \subset P_F(X)$, which satisfies the following conditions:

$i.\quad d_F((x, \lambda), (y, \gamma)) \geq 0$

$ii.\quad d_F((x, \lambda), (y, \gamma)) = 0 \ \ if \ \ and \ only \ if \ \ x = y \ \ and \ \ \lambda = \gamma = \rho = 1$

$iii.\quad d_F((x, \lambda), (y, \gamma)) = d_F((y, \gamma), (x, \lambda)) \quad [Symmetric]$

$iv.\quad d_F((x, \lambda), (z, \rho)) \leq d_F((x, \lambda), (y, \lambda)) + d_F((y, \gamma), (z, \rho)) \quad [Triangle \ \ inequlity]$

Here, d_F is a fuzzy metric in $P_F(X)$ and $d_F((x, \lambda), (y, \gamma))$ is called a fuzzy distance between the two fuzzy points.

Fuzzy linear normed space:

Suppose that L is a fuzzy linear space. The mapping $\|.\|: L \to S_F^+(\mathbb{R})$ is said to be fuzzy linear normed space for any $(x, \lambda), (y, \gamma) \in L$ if satisfies the following condition:

i) $\|(x, \lambda\| \geq 0$

ii) $\|(x, \lambda\| = 0 \; if \; and \; only \; if \; x = 0 \; and \; \lambda = 1$

iii) $\|(x, \lambda\| = |k|. \|(x, \lambda\| \; \forall \; k \in \mathbb{R}$

iv) $\|(x, \lambda) + (y, \gamma)\| \leq \|(x, \lambda\| + \|(x, \gamma\|$ [triangle inequality]

The fuzzy normed space is denoted by $(L, \|.\|)$ not that a fuzzy linear normed space L has fuzzy point belonging to the fuzzy set L as its elements.

Convergent in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space. A sequence $\{x_n\}$ in a metric space $(P_F(X), d_F)$ is said to be convergent if there exists $(x, \lambda) \in P_F(X)$ s. t. $\lim_{n \to \infty} d_F((x_n, \lambda_n), (x, \lambda)) = 0 \quad \forall \; n \in N$

Diameter in a fuzzy metric:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $S \subset P_F(X)$. The diameter of S, denoted by diam S, is defined as $diam\, S = \sup\{d_F((x, \lambda), (y, \gamma)) : (x, \lambda), (y, \gamma) \in S\}$

Cauchy sequence in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. Then the sequence $\{x_n\}$ is said to be a Cauchy sequence if for all $\varepsilon \in (0, 1]$, there exists a positive integer N s. t.

$d_F((x_n, \lambda_n), (x_m, \lambda_m)) > 1 - \varepsilon \quad \forall \; n, m \in N$

Complete in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space. An induced fuzzy metric space $(P_F(X), d_F)$ is said to be complete if any Cauchy sequence in it has a unique limit in the space.

3. Characterization of Strong and Weak convergence

Strong convergence in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. Then a sequence $\{x_n\}$ is said to be

strongly convergent if for all $\varepsilon \in (0, 1)$, there exists $(x, \lambda) \in P_F(X)$ and depending on ε s. t.

$d_F((x_n, \lambda_n), (x, \lambda)) > 1 - \varepsilon \quad \forall \; n \in N$

Weak convergence in a fuzzy metric space:

Let $(P_F(X), d_F)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in it. The maps

$d_F : P_F(X) \times P_F(X) \to S_F^+(R)$ and $f : P_F(X) \to P_F(X)$. Then a sequence $\{x_n\}$ is said to be strongly convergent if for all $\varepsilon \in (0, 1)$, there exists $(x, \lambda) \in P_F(X)$ and depending on ε s. t.

$d_F(f(x_n, \lambda_n), f(x, \lambda)) > 1 - \varepsilon \quad \forall \; n \in N$

1. Theorem (Weak convergence): Let $\{x_n\}$ be a weakly convergent sequence in a fuzzy metric space $P_F(X)$.Then

(a) The weak limit of $\{x_n\}$ is unique.

(b) Every subsequence of $\{x_n\}$ converges weakly to x.

Proof: (a) Suppose there are two limits of $\{x_n\}$ which are x and y. Then we have

$(x_n, \lambda_n) \to (x, \lambda) \quad \therefore f(x_n, \lambda_n) \to f(x, \lambda)$

$(y_n, \gamma_n) \to (y, \gamma) \quad \therefore f(x_n, \lambda_n) \to f(y, \gamma)$

Since $f(x_n, \lambda_n)$ is a sequence of numbers, its limit is unique.

Hence $f(x, \lambda) = f(y, \lambda)$ (1)

By the linearity and continuity of f, then we have

$f[(x, \lambda) - (y, \gamma)] = f(x, \lambda) - f(y, \gamma)$

$\Rightarrow f[(x, \lambda) - (y, \gamma)] = f(x, \lambda) - f(x, \gamma)$

$\Rightarrow (x, \lambda) - (y, \gamma) = 0$

$\Rightarrow (x, \lambda) = (y, \gamma)$

Hence the weak limit of $\{x_n\}$ is unique.

(b) This follows from the fact that $\{f(x_n)\}$ is a convergent sequence of number, converges and has the same limit as the sequence.

2. Theorem: Let $(P_F(X), d_F)$ be a fuzzy metric space. Then

a) A convergent sequence in $P_F(X)$ is bounded and its limit is unique.

b) If $(x_n, \lambda_n) \to (x, \lambda)$ *and* $(y_n, \gamma_n) \to (y, \gamma)$ *in* $P_F(X)$,

then $d_F((x_n, \lambda_n), (y_n, \gamma_n)) \to d_F((x, \lambda), (y, \gamma))$

Proof: Let $\{x_n\}$ be a convergence sequence in $P_F(X)$ which converges to x. Then

$\lim_{n\to\infty} d_F((x_n, \lambda_n), (x, \lambda)) = 0$

a) Let $\{x_n\}$ be a convergent sequence in $P_F(X)$ which converges to x. Then

$\lim_{n\to\infty} d_F((x_n, \lambda_n), (x, \lambda)) = 0$

$\Rightarrow d_F((x_n, \lambda_n), (x, \lambda)) > 1 - \varepsilon, \quad where\ \varepsilon \in (0, 1)$

Taking $\varepsilon = \frac{1}{2}$, there exists an integer N s. t.

$d_F((x_n, \lambda_n), (x, \lambda)) > 1 - \frac{1}{2} = \frac{1}{2}, \ n > N$

For all n, then we have

$d_F((x_n, \lambda_n), (x, \lambda)) > 1 + a$

Where $a = \{d_F((x_1, \lambda_1), (x, \lambda)), \ . \ . \ . \ , d_F((x_n, \lambda_n), (x, \lambda))\}$

This shows that $\{x_n\}$ is bounded.

For the uniqueness of the limit:

Let us assume that $(x_n, \lambda_n) \to (x, \lambda)\ and\ (x_n, \lambda_n) \to (y, \gamma)\ as\ n \to \infty$ (2)

By triangle inequality we have,

$d_F((x, \lambda), (y, \gamma)) \le d_F((x, \lambda), (x_n, \lambda_n)) + d_F((x_n, \lambda_n), (y, \gamma))$

$\Rightarrow d_F((x, \lambda), (y, \gamma)) \to 0 + 0$

$\Rightarrow (x, \lambda) = (y, \gamma)$

Hence the limit of $\{x_n\}$ is unique.

b) We have,

$$d_F((x_n, \lambda_n), (y_n, \gamma_n)) \le d_F((x_n, \lambda_n), (x, \lambda),) + d_F((x, \lambda), (y, \gamma)) + d_F((y, \gamma), (y_n, \gamma_n))$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma)) \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y, \gamma), (y_n, \gamma_n))$$

(3)

Interchanging x_n *and* x, y_n *and* y *as well as* λ_n *and* λ, γ_n *and* γ

$$d_F((x, \lambda), (y, \gamma)) - d_F((x_n, \lambda_n), (y_n, \gamma_n)) \le d_F((x, \lambda), (x_n, \lambda_n)) + d_F((y_n, \gamma_n), (y, \gamma))$$
$$\Rightarrow -[d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))] \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma))$$

(4)

From (2) and (3) together imply

$$|d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))| \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma))$$

(5)

Given that

$$(x_n, \lambda_n) \to (x, \lambda) \ as \ n \to \infty \ \ \therefore d_F((x_n, \lambda_n), (x, \lambda)) \to 0$$
$$(y_n, \gamma_n) \to (y, \gamma) \ \ as \ \ n \to \infty \ \therefore d_F((y_n, \gamma_n), (y, \gamma)) \to 0$$

(6)

From (4) and (5), then we get,

$$|d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma))| \le d_F((x_n, \lambda_n), (x, \lambda)) + d_F((y_n, \gamma_n), (y, \gamma)) \to 0 + 0$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) - d_F((x, \lambda), (y, \gamma)) \to 0$$
$$\Rightarrow d_F((x_n, \lambda_n), (y_n, \gamma_n)) \to d_F((x, \lambda), (y, \gamma))$$

4. Result and Discussion

Here, we discuss a characterization of strong and weak convergence in fuzzy metric spaces related cases.

1. Problem: Suppose L is a fuzzy linear space defined in R^n. The distance between arbitrary two fuzzy points (x, λ) *and* (y, γ) on L is defined by $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \rho\})$

Where d_E is the Euclidean distance Then show that (L, d_{FE}) is a strong fuzzy metric space where L denotes the set of fuzzy points on the fuzzy set L.

Solution: Suppose L is a fuzzy linear space in R^n. The distance between arbitrary two fuzzy points $(x, \lambda), (y, \gamma) \in L$ s. t. $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \gamma\})$

Now we will prove that (L, d_{FE}) is a strong fuzzy metric space.

i. We have $d_E(x, y) \ge 0 \quad \forall\ x, y \in L$

So that $d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min(\lambda, \gamma)) \ge 0 \ \Rightarrow d_{FE}((x, \lambda), (y, \gamma)) \ge 0$

ii. Let $d_{FE}((x, \lambda), (y, \gamma)) = 0$

Now we will prove that $x = y$ *and* $\lambda = \gamma = 1$.Then we have,

$$(d_E(x, y), \min\{\lambda, \gamma\}) = 0$$
$$\Rightarrow (d_E(x, y), \min\{\lambda, \gamma\}) = (0, 1)$$
$$\Rightarrow d_E(x, y) = 0 \quad and \ \min\{\lambda, \gamma\} = 1$$
$$\Rightarrow x = y \qquad\qquad \Rightarrow \lambda = \gamma = 1$$

Conversely, Let $x = y$ *and* $\lambda = \gamma = 1$

Now we will prove that $d_{FE}((x, \lambda), (y, \gamma)) = 0$

Then we have

$d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \gamma\})$
$\Rightarrow (d_E(x, y), \min\{\lambda, \gamma\}) = (0, 1) = 0$
$\Rightarrow d_{FE}((x, \lambda), (y, \gamma)) = 0$

iii. Since (L, d_E) is a metric space.

$\therefore \quad d_E(x, y) = d_E(y, x)$

Then we have,

$d_{FE}((x, \lambda), (y, \gamma)) = (d_E(x, y), \min\{\lambda, \gamma\}) = (d_E(y, x), \min\{\lambda, \gamma\})$
$\Rightarrow d_{FE}((x, \lambda), (y, \gamma)) = d_{FE}((y, \gamma), (x, \lambda))$

iv. Let $(x, \lambda), (y, \gamma), (z, \rho) \in L$ be arbitrary three fuzzy points.

Since (R^n, d_E) is a metric space, s. t. $d_E(x, z) \le d_E(x, y) + d_E(y, z)$ (7)

If $\lambda \in F$, *then* $y = (1-\lambda)x + \lambda z$

Let $\alpha = \min\{\lambda, \rho\}$. Then we have $\{x, z\} \subset L_\alpha$

Since L is a fuzzy linear space, L_α is a linear subspace of R^n

i. e. $y \in L_\alpha$ s.t. $\gamma = L(y) \ge \alpha = \min\{\lambda, \rho\}$

This implies that $\min\{\lambda, \gamma, \rho\} = \min\{\lambda, \rho\}$ (8)

$\therefore d_{FE}((x, \lambda), (z, \rho)) = (d_E(x, z), \min\{\lambda, \rho\})$
$\Rightarrow d_{FE}((x, \lambda), (z, \rho)) \le (d_E(x, y) + d_E(y, z), \min\{\lambda, \gamma, \rho\})$
$\Rightarrow d_{FE}((x, \lambda), (z, \rho)) \le (d_E(x, y), \min\{\lambda, \gamma, \rho\}) + (d_E(y, z), \min\{\lambda, \gamma, \rho\})$
$\Rightarrow d_{FE}((x, \lambda), (z, \rho)) \le (d_E(x, y), \min\{\lambda, \gamma\}) + (d_E(y, z), \min\{\gamma, \rho\})$
$\Rightarrow d_{FE}((x, \lambda), (z, \rho)) \le d_{FE}((x, \lambda), (y, \lambda)) + d_{FE}((y, \gamma), (z, \rho))$

Therefore (L, d_{FE}) is a strong fuzzy metric space.

2. Problem: Let (X, d) be a fuzzy metric space and $r \in R^+$, then show that (X, d_1) is a metric space, Where d_1 is define by $d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(x_k,\ y_k)}{1+rd(x_k,\ y_k)}$

Solution:(i) We have $d(x_k, y_k) \ge 0 \quad \forall\, k = 1, 2, 3, \dots\dots\dots$

$\Rightarrow \dfrac{d(x_k, y_{k\_}}{1 + rd(x_k, y_k)} \ge 0$

$\Rightarrow d_1(x, y) \ge 0$

Let $d_1(x, y) = 0$ now we will prove that $x = y$

Then we have $\sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{1+rd(x_k, y_k)} = 0$

$\Rightarrow \dfrac{d(x_k, y_k)}{1 + rd(x_k, y_k)} = 0$

$\Rightarrow d(x_k, y_k) = 0$

$\Rightarrow |x_k - y_k| = 0$

$\Rightarrow x_k - y_k = 0$

$\Rightarrow x_k = y_k$

$\Rightarrow x = y$

Conversely let $x = y$. Now we will prove that $d_1(x, y) = 0$

$\Rightarrow x_k = y_k$

$\Rightarrow x_k = y_k$

$\Rightarrow |x_k - y_k| = 0$

$\Rightarrow d(x_k, y_k) = 0$

$\Rightarrow \dfrac{d(x_k, y_k)}{1 + rd(x_k, y_k)} = 0$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{d(x_k, y_{k\_}}{1 + rd(x_k, y_k)} = 0$$

$\Rightarrow d_1(x, y) = 0$

So $d_1(x, y) = 0 \Leftrightarrow x = y$

iii) Since (X, d) is a matrix space, so $d(x_k, y_k) = d(x_k, y_k)$

Then we have $d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{1+rd(x_k, y_k)} = \sum_{k=1}^{\infty} \frac{d(y_k, x_k)}{1+rd(y_k, x_k)}$

$d_1(x, y) = d_1(y, x)$

iv) Define $f: R^+ \longrightarrow R^+$ by $f(t) = \frac{t}{1+rt}(t\epsilon R^+)$

$$f'(t) = \frac{1 + rt - rt}{(1 + rt)^2} = \frac{1}{(1 + rt)^2}$$

So, f is strictly increasing on R^+, Also, we have $d_1(x, y) = d(x_k, z_k) + d(z_k, y_k)$

This means $d_1(x, y) = \sum_{k=1}^{\infty} \frac{d(x_k, y_k)}{1+rd(x_k, y_k)} \leq \sum_{k=1}^{\infty} [\frac{d(x_k, z_k)+d(z_k, y_k)}{1+r(d(x_k, z_k)+d(z_k, y_k))}]$

$$\leq \sum_{k=1}^{\infty} \frac{d(x_k, z_k)}{1 + rd(x_k, z_k)} + \sum_{k=1}^{\infty} \frac{d(z_k, y_k)}{1 + rd(z_k, y_k)}$$

$\leq d_1(x, z) + d_1(z, y)$

$\Rightarrow d_1(x, y) \leq d_1(x, z) + d_1(z, y)$

Hence, $d_1(x, y)$is a matrix space with the given meric.

3. Problem: Let $(L, \|.\|)$ be a fuzzy linear normed space. Prove that the fuzzy norm in an inner products space L satisfies the following axioms.

i) $\|(x, \lambda)\| \geq 0$

ii) $\|(x, \lambda)\| = 0 \text{ if and only if } x = 0 \text{ and } \lambda = 1$

iii) $\|(x, \lambda)\| = |k|.\|(x, \lambda)\| \,\forall\, k \in \mathbb{R}$

iv) $\|(x, \lambda) + (y, \gamma)\| \leq \|(x, \lambda)\| + \|(x, \gamma)\|$

Where $(x, \lambda), (y, \gamma) \in L$

Solution:

i) By the definition of linear product, we get

$\langle (x, \lambda), (x, \lambda) \rangle \geq 0$

$\Rightarrow \|(x, \lambda)\|^2 \geq 0$

$\Rightarrow \|(x, \lambda)\| \geq 0$

ii) Let $\|(x, \lambda)\| = 0$

Then we have

$\langle (x, \lambda), (x, \lambda) \rangle = 0$

$\Leftrightarrow \langle (x, \lambda) - 1, (x, \lambda) \rangle = 0$

$\Leftrightarrow (x, \lambda) - 1 = 0 \text{ and } (x, \lambda) = 0 \Rightarrow x = 0$

$\Leftrightarrow (x, \lambda) = 1$

$\lambda = 1$

iii) We have

$\|k\,(x, \lambda)\|^2 = \langle k(x, \lambda), k(x, \lambda) \rangle$

$= k\,\langle (x, \lambda), k(x, \lambda) \rangle$

$= k\,\overline{\langle k(x, \lambda), (x, \lambda) \rangle}$

$= k\,\langle \overline{k(x, \lambda)}, \overline{(x, \lambda)} \rangle$

$= k\bar{k}\langle \overline{(x, \lambda)}, \overline{(x, \lambda)} \rangle$

$= |k|^2 \overline{\langle (x, \lambda), (x, \lambda) \rangle}$

$= |k|^2 \langle (x, \lambda), (x, \lambda) \rangle$

$= |k|^2 \|(x, \lambda)\|^2$

$\Rightarrow \|k\,(x, \lambda)\| = |k| \|x, \lambda\|$

iv) We have $\|(x,\lambda)+(y,\lambda)\|^2=\langle(x,\lambda)+(y,\gamma),(x,\ \lambda)+(y,\gamma)\rangle$

$=\langle(x,\lambda),(x,\lambda)+(y,\gamma)\rangle+\langle(y,\gamma),(x,\gamma)+(y,\gamma)\rangle$

$=\overline{\langle(x,\lambda)+(y,\gamma),(x,\lambda)\rangle}+\overline{\langle(x,\gamma)+(y,\gamma),(y,\gamma)\rangle}$

$=\langle\overline{(x,\lambda)+(y,\gamma)},\overline{(x,\lambda)}\rangle+\langle\overline{(x,\gamma)+(y,\gamma)},\overline{(y,\gamma)}\rangle$

$=\langle\overline{(x,\lambda)}+\overline{(y,\gamma)},\overline{(x,\lambda)}\rangle+\langle\overline{(x,\gamma)}+\overline{(y,\gamma)},\overline{(y,\gamma)}\rangle$

$=\langle\overline{(x,\lambda)},\overline{(x,\lambda)}\rangle+\langle\overline{(y,\gamma)},\overline{(x,\lambda)}\rangle+\langle\overline{(x,\lambda)},\overline{(y,\gamma)}\rangle+\langle\overline{(y,\gamma)},\overline{(y,\gamma)}\rangle$

$=\langle\overline{(x,\lambda),(x,\lambda)}\rangle+\langle\overline{(y,\gamma),(x,\lambda)}\rangle+\langle\overline{(x,\lambda),(y,\gamma)}\rangle+\langle\overline{(y,\gamma),(y,\gamma)}\rangle$

$=\langle(x,\lambda),(x,\lambda)\rangle+\langle(y,\gamma),(x,\lambda)\rangle+\overline{\langle(x,\lambda),(y,\gamma)\rangle}+\langle(y,\gamma),(y,\gamma)\rangle$

$=\|(x,\lambda)\|^2+2\,Re\,\langle(x,\lambda),(y,\gamma)\rangle+\|(y,\gamma)\|^2$

$\Rightarrow\ \|(x,\lambda)+(y,\lambda)\|^2=\|(x,\lambda)\|^2+2\ |\langle(x,\lambda),(y,\gamma)\rangle|+\|(y,\gamma)\|^2$ (9)

For Cauchy- Schwartz inequality (10)

We have $|\langle(x,y)\rangle|\le\|x\|\|y\|$

Form (9), then we get, $\|(x,\lambda)+(y,\lambda)\|^2\le\|(x,\lambda)\|^2+2\ \|(x,\lambda)\|\|(y,\gamma)\|+\|(y,\gamma)\|^2$

$\Rightarrow\ \|(x,\lambda)+(y,\lambda)\|^2\le(\|(x,\lambda)\|+\|(y,\gamma)\|)^2$

$\Rightarrow\|(x,\lambda)+(y,\lambda)\|\le\|(x,\lambda)\|+\|(y,\gamma)\|$

Lemma:

Any subsequence of a Cauchy sequence of fuzzy points is also a Cauchy sequence and has the same limit as the original one.

Proof: It is obvious from Definition of Cauchy sequence in a fuzzy metric space

Proposition: Suppose $\left(L,\|.\|_{FE}\right)$ is a fuzzy linear normed space defined in R^n. For any $(x,\lambda)\in L$

Such that $\langle(x,\lambda),(x,\lambda)\rangle=\|(x,\lambda)\|^2{}_{FE}$, where the inner product is defined in the sense of Xia and Guo (2003) [11], i. e. $\langle(x,\ \lambda),(y,\ \gamma)\rangle=(\langle x,\ y\rangle,\ \min\{\lambda,\ \gamma\})$

Proof: From the definition of inner product of fuzzy points, one has

$\langle(x,\ \lambda),(x,\ \lambda)\rangle=(\langle x,\ x\rangle,\ \lambda)$

$\Rightarrow\langle(x,\ \lambda),(x,\ \lambda)\rangle=(\|x\|^2{}_E,\ \lambda)$

$\Rightarrow\langle(x,\ \lambda),(x,\ \lambda)\rangle=(\|x\|_E,\ \lambda)(\|x\|_E,\ \lambda)$

$\Rightarrow\langle(x,\ \lambda),(x,\ \lambda)\rangle=\|(x,\ \lambda)\|^2{}_{FE}$

Conclusion

In this paper, we introduce and study a concept of strong and weak convergence in fuzzy metric spaces $(P_F(X),d_F)$ such that $P_F(X)$ is a set. Then, every fuzzy metric space can induce a fuzzy topology space, which implies in another way that the fuzzy measure defined in this paper is not only reasonable but also Significance. A natural continuation of this paper is to investigate for which of these concepts one can get a characterization of the corresponding completeness by means of certain classes of nested sequences of sets. We hope that this work will be useful for fuzzy metric space related to normed spaces. All expected results in this paper will help us to understand better solution of different fuzzy metric space related theorem. In future, we will discuss of strong and weak convergence in fuzzy metric space properties related to physical problems.

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Author contributions

Authors have made equal contributions for paper.

CONFLICTS OF INTEREST

There are no conflicts to declare.

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