icrrd
Quality Index Research Journal
INTERNATIONAL CENTER FOR RESEARCH AND RESOURCES DEVELOPMENT
article
ISSN Number: 2773-5958, SSM Number: (1347468-T), doi.org/10.53272/icrrd, www.icrrd.com

Estimates of First Two Coefficients of Analytic Functions of a Certain Class of Bi-Univalent Functions using OPPLA Differential Operator

Gumeet Singh
GSSDGS Khalsa College, Patiala, India
**Corresponding author; Email: Meetgur111@gmail.com*

Received: 10 March 2022
Revision: 20 May 2022
Accepted: 25 June 2022
Available Online: 28 July 2022
Published: XX XX 2022
Volume-3, Issue-2
Cite This: *ICRRD Qual. Ind. Res. J.* 2022, 3(2), 000-000

ABSTRACT: In this paper, we have used Opoola Differential Operator, a generalization of Sãlãgean Differential Operator and Al-Oboudi Differential Operator to define a new subclass of analytic and bi-univalent functions using quasi-subordination principle. We have obtained upper estimates for the first two coefficients of functions in the this subclass by means of Ma-Minda functions.

Mathematics Subject Classification: Primary: 30C45; Secondary: 30C50.

Keywords: Analytic and bi-univalent functions, Opoola differential operator, Coefficient bounds, Ma-Minda functions.

1. Introduction

Let A denote the class of functions analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. $f(z)$ is said to be in the class S if $f \in A$ and $f(z)$ is univalent such that $f(z)$ has the following normalization

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

with the conditions $f(0) = 0,\ f'(0) = 1.$ [1]

S is called the class of normalized univalent functions.

Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U}$, then $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ written as $f(z) \prec g(z)$ if there exist a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0) = 0,\ |\omega| < 1$ which is called the Schwarz function such that $f(z) = g(\omega(z))$. If the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z),\ z \in \mathbb{U} \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For functions $f(z)$ and $g(z)$ analytic in the unit disk $\mathbb{U}$, the function $f(z)$ is quasi-subordinate to $g(z)$ written as $f(z) \prec_q g(z)$ if there exist analytic function φ and ω with

$|\varphi(z)| \le 1,\ \ |\omega(z)| < 1$ such that

$f(z) = \varphi(z) g\big(\omega(z)\big).$ [2]

Bi-univalent functions in class S: Since univalent functions are one-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$.

The Koebe $\frac{1}{4}$ theorem ensures that the image of $\mathbb{U}$ under every univalent function $f \in S$ contain a disk of radius $\frac{1}{4}$. Thus, every univalent function f has an inverse f^{-1} satisfying $f^{-1}\big(f(z)\big) = z,\ (z \in \mathbb{U})$ and $f^{-1}\big(f(\omega)\big) = \omega,\quad |\omega| < \gamma_0(f),\ \gamma_0(f) \geq \frac{1}{4})$. Therefore, a function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent functions in $\mathbb{U}$ and it is denoted by $\sum$. [3]

The Taylor-Macluarin series of $f^{-1}(z)$ is used to find the inverse of coefficients of $f^{-1}(z)$. For

$$\omega = f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

$$z = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + (14a_2^4 - 21a_2^2a_3 + 3a_3^2 + 6a_2a_4 - a_5)\omega^5$$

In 2017, [4] considered the subclass $S_q^*\sum(n,\lambda,\gamma,\phi(z))$ of analytic and bi-univalent function associated with S$\tilde{a}$l$\tilde{a}$gean differential operator consisting of the functions of class $\sum$ in the open unit disk which satisfies the quasi-subordination conditions and the first two coefficients bounds for functions in this class were obtained by the author. Several other authors that have worked in this area included [5], [6], [7], [8] and [9] but not limited to these one only.

1. Preliminary

Lemma 1. [10]: Let $\varphi(z)$ be an analytic function with positive real part in $\mathbb{U}$, with $|\varphi(z)| \leq 1$ and $\varphi(z) = P_0 + P_1z + P_2z^2 + P_3z^3 + \cdots$ then $|P_n| \leq 1 - |P_0|^2 \leq 1$, for $n > 0$.

Let $u(z) = c_1z + c_2z^2 + c_3z^3 + \cdots$ and $v(z) = d_1z + d_2z^2 + d_3z^3 + \cdots$ be two analytic functions in $\mathbb{U}$ with the conditions $u(0) = 0, v(0) = 0, |u(z)| < 1\ and\ |v(z)| < 1$. It is well known that

$$|c_n| = |d_n| \leq \begin{cases} 1, & n = 1 \\ 1 - |c_1|^2, & n \geq 2. \end{cases}$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1,\ \phi'(0) > 0$ which maps the open unit disk $\mathbb{U}$ onto a region star-like with respect to 1 and it's symmetry with respect to the real axis. The Taylor's series of such functions is given by $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots,\ where\ B_1 > 0$. [11]

Opoola Differential Operator. [12]: For $\wp \geq 0, 0 \leq \mu \leq \beta, n \in \mathbb{N}_0, z \in \mathbb{U}$. the Opoola differential operator $D^n(\wp,\mu,\beta)f: A \to A$ is defined as follows

$$D^0(\wp,\mu,\beta)f(z) = f(z)$$

$$D^1(\wp,\mu,\beta)f(z) = \wp z f'(z) - z(\beta-\mu)\wp + (1 + (\beta-\mu-1)\wp)f(z)$$

$$D^n(\wp,\mu,\beta)f(z) = (D(D^{n-1}(\wp,\mu,\beta)f(z))) = z + \sum_{k=2}^{\infty}[1 + (k+\beta-\mu-1)\lambda]^n a_k\, z^k \qquad (2)$$

Remarks:

- When $\wp = 1,\ \mu = \beta,$ then $D^n(\wp,\mu,\beta)f(z)$ becomes the Salagean differential operator. [13]
- When $\mu = \beta,$ then $D^n(\wp,\mu,\beta)f(z)$ becomes the Al-Oboudi differential operator. [14]

This operator was studied by [15] and [16].

1. Main Result

Definition: A function $f(z) \in \sum$ belongs to the class $N_q\sum(\alpha,n,\lambda,\gamma,\mu,\beta,t,\phi(z))$ if

$$\frac{1}{\gamma}\left\{\frac{z[(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)]'}{(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)}\right\} \prec_q \left(\phi(z)\right)^\alpha - 1, \quad z\in \mathbb{U}. \tag{3}$$

and

$$\frac{1}{\gamma}\left\{\frac{\omega[(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)]'}{(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)}\right\} \prec_q (\phi(\omega))^\alpha - 1, \quad \omega\in \mathbb{U}. \tag{4}$$

$0\le\lambda\le 1, n\in N_0, \gamma\in\mathbb{C}/0, g(\omega)=f^{-1}(\omega),\ D^n(\mu,\beta,t)f(z)$ is the Opoola differential operator defined in equation (3).

Remarks:

1. When $\alpha=\mu=\beta=t=1,\ N_q\sum(\alpha,n,\lambda,\gamma,\mu,\beta,t,\phi(z))\equiv SC_q,\sum(n,\lambda,\gamma,\phi(z))$. See[4].
2. When $n=0,\ \ \alpha=\mu=\beta=t=\gamma=\lambda=1,\ N_q\sum\left(\alpha,n,\lambda,\gamma,\mu,\beta,t,\phi(z)\right)\equiv C_q,\sum\left(\gamma,\phi(z)\right)$. see[9].

Theorem 1. If the function $f(z)\in A$ given by (1) be in the class $N_q,\sum\left(\alpha,n,\lambda,\gamma,\mu,\beta,t,\phi(z)\right)$, then

$$|a_2|\le\sqrt{\frac{\alpha|\gamma|[|B_1|+|B_2|+(\alpha-1)B_1^2]}{[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}-\{[1+(\beta-\mu-1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2\}}}$$

$$|a_3| \le \frac{\alpha\gamma B_1}{2[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}} + \frac{2\alpha|\gamma|[|B_1|+|B_2|+(\alpha-1)B_1^2]}{[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}-2[1+(\beta-\mu+1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2}$$

Proof: Since $f(z)\in N_q\sum\left(n,\lambda,\gamma,\mu,\beta,t,\phi(z)\right)$, there exist two analytic functions $u,v\colon U\to V\ \ with\ u(0)=0, v(0)=0$ such that

$$\frac{1}{\gamma}\left\{\frac{z[(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)]'}{(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)}\right\}=\varphi(z)[\{\phi(u(z))\}^\alpha-1]$$

and

$$\frac{1}{\gamma}\left\{\frac{\omega[(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)]'}{(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)}\right\}=\varphi(\omega)[\{\phi(u(\omega))\}^\alpha-1]$$

From lemma 1 and [11], $\quad u(z)=c_1z+c_2z^2+c_3z^3+\cdots,$

$$v(\omega)=d_1\omega+d_2\omega^2+d_3z^3+\cdots$$

$$\phi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots,\qquad \phi(\omega)=1+B_1\omega+B_2\omega^2+B_3\omega^3+\cdots$$

$$\varphi((z)=p_0+p_1z+p_2z^2+p_3z^3+\cdots,\qquad \varphi((\omega)=p_0+p_1\omega+p_2\omega^2+p_3\omega^3+\cdots$$

Then,

$$\varphi(z)\left[\{\phi(u(z))\}^\alpha-1\right] = \alpha p_0B_1c_1z+\alpha\left[p_1B_1c_1+p_0\left(B_1c_2+B_2c_1^2+\frac{\alpha-1}{2}B_1^2c_1^2\right)\right]z^2+\cdots \tag{5}$$

And

$$\varphi(\omega)\left[\{\phi(u(\omega))\}^{\alpha}-1\right] = \alpha p_0 B_1 d_1 \omega + \alpha\left[p_1 B_1 d_1 + p_0\left(B_1 d_2 + B_2 d_1^2 + \frac{\alpha-1}{2}B_1^2 d_1^2\right)\right]\omega^2 + \cdots \qquad (6)$$

The left hand side of equations (3) and (4) can be expressed as

$$\frac{1}{\gamma}\left\{\frac{z[(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)]'}{(1-\lambda)D^n(\mu,\beta,t)f(z)+\lambda D^{n+1}(\mu,\beta,t)f(z)}\right\}$$
$$=\frac{1}{\gamma}\left\{\frac{\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k z^{k-1}\{k-1+\lambda(1-k)+[1+(+(k+\beta-\mu-1)t](\lambda k-\lambda)\}}{1+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k z^{k-1}\{1-\lambda+\lambda[1+(+(k+\beta-\mu-1)t]\}}\right\} \qquad (7)$$

And

$$\frac{1}{\gamma}\left\{\frac{\omega[(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)]'}{(1-\lambda)D^n(\mu,\beta,t)f(\omega)+\lambda D^{n+1}(\mu,\beta,t)f(\omega)}\right\}=$$
$$=\frac{1}{\gamma}\left\{\frac{\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k \omega^{k-1}\{k-1+\lambda(1-k)+[1+(+(k+\beta-\mu-1)t](\lambda k-\lambda)\}}{1+\sum_{k=2}^{\infty}[1+(k+\beta-\mu-1)t]^n a_k \omega^{k-1}\{1-\lambda+\lambda[1+(+(k+\beta-\mu-1)t]\}}\right\} \qquad (8)$$

Comparing the coefficients of the like powers of z in (5), (6), (7) and (8), then

$$\frac{1}{\gamma}\{[1+(\beta-\mu-1)t]^n\{1+\lambda t(\beta-\mu+1)\}a_2\} = \alpha p_0 B_1 c_1 \qquad (9)$$

$$\frac{1}{\gamma}\{[1+(\beta-\mu-2)t]^n\{2+2\lambda t(\beta-\mu+2)\}a_3\}-\{[1+(\beta-\mu-1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2 a_2^2\}$$
$$=\alpha\left[p_1 B_1 c_1 + p_0\left(B_1 c_2 + B_2 c_1^2 + \frac{\alpha-1}{2}B_1^2 c_1^2\right)\right] \qquad (10)$$

$$-\frac{1}{\gamma}\{[1+(\beta-\mu-1)t]^n\{1+\lambda t(\beta-\mu+1)\}a_2\} = \alpha p_0 B_1 d_1 \qquad (11)$$

$$\frac{1}{\gamma}\{[1+(\beta-\mu-2)t]^n\{2+2\lambda t(\beta-\mu+2)\}(2a_2^2-a_3)\}-\{[1+(\beta-\mu-1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2 a_2^2\} = \alpha\left[p_1 B_1 d_1 + p_0\left(B_1 d_2 + B_2 d_1^2 + \frac{\alpha-1}{2}B_1^2 d_1^2\right)\right] \qquad (12)$$

It follows from equations (9) and (11) that

$$c_1 = -d_1 \qquad (13)$$

Using equations (13) in (10) and (12), one can easily obtain

$$|a_2^2| = \frac{\alpha\gamma p_0[B_1(c_2+d_2)+2B_2 d_1^2+(\alpha-1)B_1^2 d_1^2]}{2[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}-2\{[1+(\beta-\mu-1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2\}} \qquad (14)$$

On applying lemma 1, one can easily have

$$|a_2| \le \sqrt{\frac{\alpha|\gamma|[|B_1|+|B_2|+(\alpha-1)B_1^2]}{[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}-\{[1+(\beta-\mu-1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2\}}} \qquad (15)$$

Hence we get the desired bound on $|a_2|$

To get the bound on $|a_3|$, using (14) in (10) and (12), we have

$$a_3 = \frac{\alpha\gamma[-2p_1B_1d_1 + p_0B_1(d_2 - c_2)]}{2[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}} + \frac{\alpha\gamma p_0[B_1(c_2+d_2) + 2B_2d_1^2 + (\alpha-1)B_1^2d_1^2]}{2[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\} - 2[1+(\beta-\mu+1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2}$$

Using lemma 1, bound on a_3 is obtained as

$$|a_3| \le \frac{\alpha\gamma B_1}{2[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\}} + \frac{2\alpha|\gamma|[|B_1| + |B_2| + (\alpha-1)B_1^2]}{[1+(\beta-\mu+2)t]^n\{2+2\lambda t(\beta-\mu+2)\} - 2[1+(\beta-\mu+1)t]^{2n}\{1+\lambda t(\beta-\mu+1)\}^2} \quad (16)$$

Remarks:

1. When $\alpha = \beta = \mu = t = 1$, the inequalities in (15) and (16) reduced to the results of theorem 2 in [4].
2. When $n = 0,\ \lambda = 0,\ \ \alpha = \gamma = 1$, the inequalities in equations (15) and (16) reduced to the result of theorem 2.5 in [9].

Conclusion: A new subclass of univalent and bi-univalent functions was defined by means of quasi-subordination principle and the result obtained is a generalization of the results obtained in [4] and [9].

REFERENCES

[1] Duren, P.L.: Univalent Functions. *Springer, New York, NY. USA. 1983.*

[2] Miller, S.S, Mocanu, P.T.: Differential Subordination. *Theory and Applications, Marcel-Dekker Inc., New York;* 2000.

[3] Koebe, P.: "Uber die Uniformisierung beliebiger analytischer Kurven". *Math.Phys. Kl*. (1907), 191-210, 633-669.

[4] Ramachandran, C, Kavitha, D.: Coefficient estimates for a subclass of bi-univalent functions defined by Salagean operator using quasi-subordination. *Appplied mathematical sciences*. Vol.11, 2017, no.35,1725-1732.

[5] Brannan, D.A, Taha, T.S.: On some classes of bi-univalent functions. *Studia Univ.Babes,Bolyai Math.,* 31(1986). no.2, 70-77.

[6] Moh'd, M.H, Darus, M.: Fekete-Szego problems for quasi-subordination classes. *Abstract and Applied Analysis*. vol. 2012(2012). Article ID 192956.

[7] Cağlar, M., Orhan, H, Yagmur, N.: Coefficient bounds for new subclasses ofbi-univalent functions, *Filomat,* 27(2013).

[8] Deniz, E.: Certain subclasses of bi-univalent functions satisfying subordinate conditions. *J. Classs. Anal.*, 2(2013), 49-60.

[9] Obaid, A.: Estimates of initial coefficient for certain subclasses of bi-univalent functions involving quasi-subordination. *Journal of Nonlinear science and Applications*. 10(2017), 1004-1011.

[10] Nehari, Z.: *Conformal Mapping. Dover, New York, NY, USA. 1975. Reprinting of the 1952 edition.*

[11] Ma, W, Minda, C.D.: A unified treatment of some special classes of functions. *Proceedings of the Conference on complex analysis.* 1994.

[12] Opoola, T.O.: On a Subclass of Univalent Functions Defined by a Generalized Differential Operator, International *Journal of Mathematical Analysis.*2017; 11(18): 869-876. https:doi.org/10.12988/ijma.2017.7252.

[13] Salagean, GS. Subclasses of Univalent Functions. Complex Analysis. *Fifth Romanian Seminar, Lecture Notes in Mathematics.* 1983; 1013: Springer,Berlin. 362-372.

[14] Al-Oboudi, FM. On univalent functions defined by a generalized Salagean operator. *Int.J. math.sci.* (25-28)2004; 1429-143.

[15] Fatunsin, LM, Opoola, TO. New results on subclasses of analytic functions defined by Opoola differential operator. *Journal of Mathematical Science System*. 7(2017), 289-295.

[16] Oyekan, EA,Swamy, SR, Opoola, TO. Rushewey derivative and a new generalized operator involving convolution. *International journal of mathematics trends and technology*. vol.67, issue1.